Distributed Optimization of Multi-Agent Systems over Uniform Hypergraphs

Yaqi Hao, Ji-Feng Zhang, *Fellow, IEEE*

***Abstract*— Distributed optimization of multi-agent systems over uniform hypergraphs is considered in this paper. Given a global objective function in advance, an ideal design method of utility functions for each agent is put forward to convert a multi-agent system into a budget-balanced potential network game (BBPNG) with the pre-assigned objective function as its potential function. First, the verification of BBPNGs is simplified to verify whether its fundamental network game is a budget-balanced potential game (BBPG). Next, the algebraic and geometric expressions of BBPGs are obtained, respectively. Finally, a necessary and sufficient condition is given about the utility design.**

***Index Terms*— Potential network game, budget balance condition, Nash equilibrium, semi-tensor product of matrices.**

I. Introduction

ONE key feature in the optimization problem is the existence of a system objective function that the system designer seeks to optimize, such as the resource allocation problem [12], the sensor coverage problem [16], etc. Distributed algorithms aim to optimize this function effectively through the local information exchange between individuals with the help of the network [1], [3]. Due to the rationality of individuals, the final state of the system is often determined by the mutual influence and interaction between individuals' strategies, so it is necessary to design individuals' optimal strategies to achieve the system optimization. Finite non-cooperative game precisely studies the strategy selection and optimization among individuals with conflicting interaction relation, so it plays an important role in establishing the analysis model of the system optimization [14], [15]. Particularly, finite potential games are favored because of the existence and the convergence of Nash equilibria [13] since Nash equilibria are used to correspond to the system's equilibrium behaviors, i.e., the states that improve the system performance. Potential based optimization becomes a promising method for the system optimization [7], [9].

After modeling the system as a finite game, each individual will be assigned a utility function. Since an individual's choice of the strategy is based on the utility obtained under the strategy, it is reasonable to induce individuals to choose optimal strategies by designing proper utility functions. Generally, the system objective function equals to the sum of individuals' utility functions in most all distributed optimization problems. For example, in the study of the control (or influence) of the social system, it describes the phenomenon that there are costs or benefits that need to be fully absorbed by participants [12].

[6] puts forward a solution framework for the system optimization which takes potential games as the interface of two separate design steps, one of which is the utility design aiming to convert the system into a finite potential game. However, there may be multiple Nash equilibria, not all of which can improve the system performance. A new technology for the utility design is proposed in this paper, which takes the optimality of Nash equilibria into account. Given that the optimal value point of the potential function is also a Nash equilibrium, which is called the effective (potential-maximizing\minimizing) Nash equilibrium, the system objective function is further taken as the potential function. In this way, the optimal equilibrium behavior of the system naturally corresponds to the effective Nash equilibrium of the finite potential game, and then, its existence can be guaranteed.

This paper studies the optimization of multi-agent systems over uniform hypergraphs via potential game approach. In terms of the system optimization, most researches are carried out on network graphs based on binary relationships [11]. However, for high-order interactions in social and communication networks, only the hypergraph can describe them well. Particularly, the uniform hypergraph is an important part of the hypergraph theory [2]. So far, the research under hypergraphs has not received much attention. In terms of the finite potential game theory, there have been fruitful results [8], [10]. However, taking the system objective function as the potential function means that the potential function equals to the sum of all players' utility functions (we refer to this as the budget balance condition). This is a new type of finite potential games, which has not yet been proposed and studied. Hence, there are three innovative results, specifically: 1) the verification of budget-balanced potential network games (BBPNGs); 2) algebraic and geometric expressions of budget-balanced potential games (BBPGs); 3) a necessary and sufficient condition for the utility design.

The work was supported by the National Key R&D Program of China under Grant 2018YFA0703800, and by the National Natural Science Foundation of China under Grant T2293770. This work was done when the first author as a postdoctor worked at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. (Corresponding author: Ji-Feng Zhang.)

Yaqi Hao is with School of Mathematics, Shandong University, Shandong 250199, China (e-mail: hoayaqi@outlook.com).

Ji-Feng Zhang is with Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Beijing 100190, China; Chinese Academy of Sciences and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100149, China (e-mail: jif@iss.ac.cn).

The rest of this paper is organized as follows. Section 2 introduces the semi-tensor product of matrices. Section 3 discusses the verification of BBPNGs and algebraic and geometric structures of BBPGs. Section 4 gives the utility design method. Section 5 concludes the paper.

For statement ease, we first give some notations: (i) The set of $m \times n$ real matrices is denoted by $\mathcal{M}_{m\times n}$. (ii) $\mathbf{1}_\ell$ is a $\ell-$dimensional column vector with all entries equal to 1, and $\mathbf{0}_{p\times q}$ is a $p \times q$ matrix with zero entries. (iii) $\mathrm{Col}(M)$: the set of columns of $M \in \mathcal{M}_{m\times n}$.

II. Preliminaries

The semi-tensor product of matrices is defined as follows:

Definition 2.1: [4] Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$. The semi-tensor product of matrices of M and N is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p},$$

where $t = lcm(n, p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Throughout this paper the default matrix product is the semi-tensor product, that is $AB := A \ltimes B$.

Next, we consider the matrix expression of logical relations. Let $\mathcal{D}_k := \{1, 2, \cdots, k\}$ be a finite set with k logical variables, in which the concrete number j is used to represent the $j-$th element in the set. Define the vector form expression of a logical variable $j \in \mathcal{D}_k$ as $\vec{j} := \delta_k^j \in \Delta_k$. We identify $j \in \mathcal{D}_k$ to $\delta_k^j \in \Delta_k$, i.e., $j \sim \delta_k^j$, $j = 1, 2, \cdots, k$. Here, δ_k^i is the $i-$th column of identity matrix I_k, and $\Delta_k := \{\delta_k^i \mid i = 1, 2, \cdots, k\}$ is the set of all columns.

Proposition 2.2: [4] For a function $f : \prod_{i=1}^n \mathcal{D}_{k_i} \to R$, there exists a unique $V_f \in \mathbb{R}^k$ with $k = \Pi_{i=1}^n k_i$, such that (in vector form) we have

$$f(x_1, \cdots, x_n) = V_f \ltimes_{i=1}^n \vec{x}_i, \tag{1}$$

where $x_i \in \mathcal{D}_{k_i}$, $i = 1, 2, \cdots, n$; $\ltimes_{i=1}^n \vec{x}_i := \vec{x}_1 \ltimes \vec{x}_2 \ltimes \cdots \ltimes \vec{x}_n$. Moreover, V_f is called the structure vector of f.

Example 2.3: Assume $n = 2$, $k_1 = k_2 = 2$, and $\mathcal{D}_2 = \{1, 2\}$. Define a function $f : \mathcal{D}_2\times\mathcal{D}_2 \to \mathbb{R}$ as below: $f(1, 1) = a_1$, $(1, 2) = a_2$, $f(2, 1) = b_1$, $f(2, 2) = b_2$. Then, $V_f = [a_1, a_2, b_1, b_2]$. Let $1 \sim \vec{1} = [1, 0]^{\mathrm{T}}$, $2 \sim \vec{2} = [0, 1]^{\mathrm{T}}$. Then, it follows from (1) that

$$f(1, 2) = V_f \ltimes \vec{x}_1 \ltimes \vec{x}_2 = V_f \ltimes [1, 0]^{\mathrm{T}} \ltimes [0, 1]^{\mathrm{T}} = a_2.$$

III. Budget-Balanced Potential Network Games

This section introduces network games with multiple types of players, and studies the algebraic and geometric properties of BBPNGs.

A. Network Games with Multiple Types of Players

A (normal form non-cooperative) finite game $G = (N, (S_i)_{i\in N}, (c_i)_{i\in N})$ consists of three ingredients: $N := \{1, 2, \cdots, n\}$ is the set of finite players. $S_i := \{1, 2, \cdots, k_i\}$ is the set of finite strategies of player i, $i = 1, 2, \cdots, n$. A strategy combination $s = (s_1, s_2, \cdots, s_n)$ is called a strategy profile, where $s_i \in S_i$ is the strategy that player i takes. Denote by $S := \prod_{i=1}^n S_i$ the set of strategy profiles. $c_i : S \to \mathbb{R}$ is the utility function of player i, $i = 1, 2, \cdots, n$. Denote by $\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$ the set of all finite games with $|N| = n$, $|S_i| = k_i$, $i = 1, 2, \cdots, n$.

Denote the strategy profile of players other than player i as $s_{-i} := \{s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n\} \in S_{-i} = \prod_{j\in N\setminus\{i\}} S_j$. With this notation, denote $s = (s_i, s_{-i})$, and $c_i(s) = c_i(s_i, s_{-i})$.

An k-uniform hypergraph is a pair $(V, \mathcal{E})$, where $V = \{v_1, v_2, \cdots, v_n\}$ is a finite set of n vertices (nodes), and $\mathcal{E} \subset \underbrace{N \times N \times \cdots N}_{n}$ is the set of hyperedges with each hyperedge $e \in \mathcal{E}$ a finite set of containing k nodes. In Fig. 1, we give an example of a 3$-$uniform hypergraph with $V = \{1, 2, 3, 4, 5\}$ and $\mathcal{E} = \{e_1, e_2, e_3\} = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_1, v_4, v_5\}\}$.

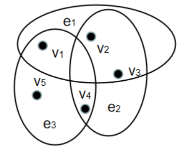

Fig. 1. An example of a 3-uniform hypergraph

A network game with multiple types of players is represented by a quadruple $G = (T, (N, \mathcal{E}), g, (c_i)_{i\in N})$, where

1) $T := \{t_1, t_2, \cdots, t_m\}$ is the set of finite types. Each player belongs to a certain type. Denote by T_i the type of player i. Depending on types, N can be divided into m disjoint sets as below:

$$N_{t_j} := \{i \in N | T_i = t_j\}, \quad j = 1, 2, \cdots, m.$$

2) $(N, \mathcal{E})$ is an $m-$uniform hypergraph. $\mathcal{E} \subset N_{t_1} \times N_{t_2} \times \cdots \times N_{t_m}$ is the set of hyperedges with each player as a node, and a hyperedge $e \in \mathcal{E}$ consists of m different types of players.
3) $g := (N_g, (S_j^g)_{j\in N_g}, (c_j^g)_{j\in N_g})$ is the fundamental network game (FNG). It is a finite m-player game, and a player in FNG is actually a type. Then, we have $N_g = T$. Assume players of the same type have the same number of strategies, and denote by $S_j^g := \{1, 2, \cdots, k_{t_j}\}$ the strategy set of players of type t_j. Denote by S^g the set of strategy profiles in the FNG. A strategy profile $s \in S^g$ is expressed as $s = (s_{t_1}, s_{t_2}, \cdots, s_{t_m})$ with s_{t_j} the strategy of type t_j. $c_j^g : S^g \to \mathbb{R}$ is the utility function for type t_j.

Denote by $\mathcal{G}_{[n;m;k_{t_1},\cdots,k_{t_m}]}$ the set of network games with multiple types of players where $|N| = n$, $|T| = m$, and $|S_i| = k_{T_i}$, $i = 1, 2, \cdots, n$. Let $|N_{t_j}| = n_j$, $j = 1, 2, \cdots, m$. Without loss of generality, we assume and set

$$N_{t_1} := \{1, 2, \cdots, n_1\};$$
$$\vdots$$
$$N_{t_m} := \{n_{m-1} + 1, n_{m-1} + 2, \cdots, n_m\}.$$

Putting all edges that contain player $i \in N$ together, we get

$$\mathcal{E}_i := \{e \in \mathcal{E} | i \in e\}, \quad i = 1, 2, \cdots, n.$$

Assume player i plays the FNG on each $e \in \mathcal{E}_i$ with the same strategy, respectively. Define the utility function of player i as the sum of utilities gained in each FNG as below:

$$c_i(s) = \sum_{e \in \mathcal{E}_i} c^g_{T_i}(s_i, s_{e\setminus\{i\}}), \quad i = 1, 2, \cdots, n, \tag{2}$$

where $s_{e\setminus\{i\}}$ is the strategy profile of those players in $e \setminus \{i\}$.

B. Verification of BBPNGs

Definition 3.1: [13] Let $G = (N, (S_i)_{i\in N}, (c_i)_{i\in N})$ be a finite game. If there exists a function $p : S \to \mathbb{R}$, such that for every $i \in N$, every $s_{-i} \in S_{-i}$, and any $s_i, s'_i \in S_i$, we have

$$c_i(s_i, s_{-i}) - c_i(s'_i, s_{-i}) = p(s_i, s_{-i}) - p(s'_i, s_{-i}), \tag{3}$$

then G is called an (exact) potential game, and the function p is called an (exact) potential function.

Definition 3.2: Let $G = (N, (S_i)_{i\in N}, (c_i)_{i\in N})$ be a finite potential game and $p : S \to \mathbb{R}$ be its potential function. If there exists a constant $c \in \mathbb{R}$, such that

$$p(s) = \sum_{i=1}^{n} c_i(s) + c, \quad \forall s \in S, \tag{4}$$

then G is called a BBPG, and (4) is called the budget balance condition. Denote by $\mathcal{G}^{\mathcal{BP}}_{[n;k_1,k_2,\cdots,k_n]}$ the set of all BBPGs with $|N| = n$, $|S_i| = k_i$, $i = 1, 2, \cdots, n$.

Remark 3.3: In this paper, we consider the case where $c = 0$ for the following reasons: 1) $c = 0$ means the total welfare of the social network system is fully distributed to all individuals with neither a surplus (positive c) nor a "blank check" (negative c); 2) the value of c does not affect the establishment of theoretical results since it is always eliminated during the proof process.

A network game with multiple types of players is called a potential network game if it is a potential game, and further called a BBPNG if its potential function satisfies the budget balance condition. Now, we discuss the relationship between a BBPNG and its FNG in two steps to simplify the verification of BBPNGs. First, consider the potential network game and its FNG. Then, take the budget balance condition into consideration.

For the verification of finite potential games, we have:

Theorem 3.4: [13] A game $G = (N, (S_i)_{i\in N}, (c_i)_{i\in N})$ is a potential one, if and only if, for every $i, j \in N$, every $a \in S_{-\{i,j\}}$, and any $x_i, y_i \in S_i$, $x_j, y_j \in S_j$, we have

$$\begin{aligned} &c_i(B) - c_i(A) + c_j(C) - c_j(B) \\ &\quad + c_i(D) - c_i(C) + c_j(A) - c_j(D) = 0, \end{aligned} \tag{5}$$

where $A = (x_i, x_j, a)$, $B = (y_i, x_j, a)$, $C = (y_i, y_j, a)$, $D = (x_i, y_j, a)$ see Figure 2.

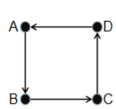

Fig. 2. A closed path of length 4

The following result is obtained according to Theorem 3.4.

Theorem 3.5: A network game with multiple types of players is a potential network game, if and only if, its FNG is a potential game.

Proof: For any two players $i, j \in N$, denote

$$\begin{aligned} \mathcal{E}_{i,j} &:= \mathcal{E}_i \cap \mathcal{E}_j; \quad \mathcal{E}_{i\setminus j} := \mathcal{E}_i \setminus \mathcal{E}_j; \\ \mathcal{E}_{j\setminus i} &:= \mathcal{E}_j \setminus \mathcal{E}_i; \quad e^- := e \setminus \{i, j\}. \end{aligned}$$

If $\mathcal{E}_{i,j} \neq \emptyset$, it can be calculated that

$$c_i(A) := \sum_{e \in \mathcal{E}_{i,j}} c^g_{T_i}(x_i, x_j, x_{e^-}) + \sum_{e \in \mathcal{E}_{i\setminus j}} c^g_{T_i}(x_i, x_{e\setminus\{i\}}).$$

Similarly, we can obtain $c_i(B)$, $c_i(C)$, and $c_i(D)$. A straightforward computation gives

$$\begin{aligned} &c_i(B) - c_i(A) + c_i(D) - c_i(C) \\ &= \sum_{e \in \mathcal{E}_{i,j}} [c^g_{T_i}(x_i, x_j, x_{e^-}) - c^g_{T_i}(y_i, x_j, x_{e^-}) \\ &\qquad + c^g_{T_i}(y_i, y_j, x_{e^-}) - c^g_{T_i}(x_i, y_j, x_{e^-})]. \end{aligned} \tag{6}$$

Similarly, for player j we have

$$\begin{aligned} &c_j(C) - c_j(B) + c_j(A) - c_j(D) \\ &= \sum_{e \in \mathcal{E}_{i,j}} [c^g_{T_j}(y_i, y_j, x_{e^-}) - c^g_{T_j}(y_i, x_j, x_{e^-}) \\ &\qquad + c^g_{T_j}(x_i, x_j, x_{e^-}) - c^g_{T_j}(x_i, y_j, x_{e^-})]. \end{aligned} \tag{7}$$

(Necessity) Consider any hyperedge $e = \{i_1, i_2, \cdots, i_m\} \in \mathcal{E}$ with $T_{i_j} = t_j$, $j = 1, 2, \cdots, m$. For any two players $i_p \in e$ and $i_q \in e$, assume all players of the same type in all hyperedges belonging to $\mathcal{E}_{i_p} \bigcap \mathcal{E}_{i_q}$ except for players i_p and i_q choose the same strategy. Then,

$$x_{e\setminus\{i_p,i_q\}} = x_{e'\setminus\{i_p,i_q\}}, \quad \forall\ e,\ e' \in \mathcal{E}_{i_p,i_q}. \tag{8}$$

Inserting (8) into (6) and (7), respectively, we have

$$\begin{aligned} &c_{i_p}(B) - c_{i_p}(A) + c_{i_p}(D) - c_{i_p}(C) \\ &= |\mathcal{E}_{i_p,i_q}|[c^g_{t_p}(x_{i_p}, x_{i_q}, x_{e^-}) - c^g_{t_p}(y_{i_p}, x_{i_q}, x_{e^-}) \\ &\qquad + c^g_{t_p}(y_{i_p}, y_{i_q}, x_{e^-}) - c^g_{t_p}(x_{i_p}, y_{i_q}, x_{e^-})]; \\ &c_{i_q}(C) - c_{i_q}(B) + c_{i_q}(A) - c_{i_q}(D) \\ &= |\mathcal{E}_{i_p,i_q}|[c^g_{t_q}(y_{i_p}, y_{i_q}, x_{e^-}) - c^g_{t_q}(y_{i_p}, x_{i_q}, x_{e^-}) \\ &\qquad + c^g_{t_q}(x_{i_p}, x_{i_q}, x_{e^-}) - c^g_{t_q}(x_{i_p}, y_{i_q}, x_{e^-})]. \end{aligned} \tag{9}$$

Substituting (9) into the left side of (5), we have

$$\begin{aligned} &c^g_{t_p}(B) - c^g_{t_p}(A) + c^g_{t_q}(C) - c^g_{t_q}(B) \\ &\quad + c^g_{t_p}(D) - c^g_{t_p}(C) + c^g_{t_q}(A) - c^g_{t_q}(D) = 0, \end{aligned}$$

implying that the FNG is a potential game.

(Sufficiency) Since the FNG is a potential game, for any two players i and j, we have

$$\begin{aligned} &c^g_{T_i}(B) - c^g_{T_i}(A) + c^g_{T_j}(C) - c^g_{T_j}(B) \\ &\quad + c^g_{T_i}(D) - c^g_{T_i}(C) + c^g_{T_j}(A) - c^g_{T_j}(D) = 0. \end{aligned} \tag{10}$$

If $\mathcal{E}_{i,j} \neq \emptyset$, then substituting (6) and (7) into the left side of (5) and using (10) yields (5), showing that the network game with multiple types of players is a potential game. Otherwise, it can be calculated that

$$c_i(A) := \sum_{e \in \mathcal{E}_i} c^g_{T_i}(x_i, x_{-e\setminus\{i\}}).$$

Similarly, we can calculate $c_i(B)$, $c_i(B)$, and $c_i(D)$. A straightforward computation gives

$$c_i(B) - c_i(A) + c_i(D) - c_i(C) = 0. \tag{11}$$

Similarly, for player j, it can also be obtained that

$$c_j(C) - c_j(B) + c_j(A) - c_j(D) = 0. \tag{12}$$

The conclusion follows immediately from (11) and (12). ■

Corollary 3.6: The potential function of a potential network game can be calculated as below:

$$p(s) = \sum_{e\in\mathcal{E}} p^g(s_{t_1}, s_{t_2}, \cdots, s_{t_m}), \tag{13}$$

where p^g the potential function of the FNG, and s_{t_j} is the strategy of the player of type t_j on the hyperedge e, $j = 1, 2, \cdots, m$.

Proof: Since the FNG is a potential game, for every $i \in N$, every $e \in \mathcal{E}_i$, and any $s_i,\ s'_i \in S_i$, we have

$$c^g_{T_i}(s_i, s_{e\backslash\{i\}}) - c^g_{T_i}(s'_i, s_{e\backslash\{i\}}) = p^g(s_i, s_{e\backslash\{i\}}) - p^g(s'_i, s_{e\backslash\{i\}}).$$

Using (13), it can be calculated that

$$\begin{aligned} & p(s_i, s_{-i}) - p(s'_i, s_{-i}) \\ = & \sum_{e\in\mathcal{E}_i} [p^g(s_i, s_{e\backslash\{i\}}) - p^g(s'_i, s_{e\backslash\{i\}})] \\ = & \sum_{e\in\mathcal{E}_i} [c^g_{T_i}(s_i, s_{e\backslash\{i\}}) - c^g_{T_i}(s'_i, s_{e\backslash\{i\}})] \\ = & c_i(s_i, s_{-i}) - c_i(s'_i, s_{-i}). \end{aligned}$$

Hence, p is the potential function of a potential network game. ■

Now, we generalize Theorem 3.5 to the budget balance case.

Theorem 3.7: A network game with multiple types of players is a BBPNG, if and only if, the FNG is a BBPG.

Proof: Put players of the same type together, and denote as

$$N_{t_j} := \{i^j_1, i^j_2, \cdots, i^j_{n_j}\}, \quad j = 1, 2, \cdots, m.$$

As the type of each player is unchanged, we have

$$N = \cup_{j=1}^m N_{t_j}, \tag{14}$$

where

$$N_{t_p} \cap N_{t_q} = \emptyset, \quad if \quad t_p \neq t_q.$$

Since each hyperedge contains m different types of players, the following fact can be verified:

$$\mathcal{E} = \cup_{t=1}^{n_j} \mathcal{E}_{i^j_t}, \quad j = 1, 2, \cdots, m, \tag{15}$$

where

$$\mathcal{E}_{i^j_p} \cap \mathcal{E}_{i^j_q} = \emptyset, \quad if \quad p \neq q.$$

(Sufficiency) Since the FNG is a potential game, the network game with multiple types of players is also a potential game. According to Corollary 3.6, it can be calculated that

$$\begin{aligned} p(s) &= \sum_{e\in\mathcal{E}} p^g(s_{t_1}, s_{t_2}, \cdots, s_{t_m}) \\ &= \sum_{e\in\mathcal{E}} \sum_{j=1}^m c^g_{t_j}(s_{t_j}, s_{e\backslash\{i\in e|T_i=t_j\}}) \\ &= \sum_{j=1}^m \sum_{e\in\mathcal{E}} c^g_{t_j}(s_{t_j}, s_{e\backslash\{i\in e|T_i=t_j\}}). \end{aligned}$$

Further, using (14) and (15), we have

$$\begin{aligned} p(s) &= \sum_{j=1}^m \sum_{t=1}^{n_j} \sum_{e\in\mathcal{E}_{i^j_t}} c^g_{t_j}(s_{i^j_t}, s_{e\backslash\{i^j_t\}}) \\ &= \sum_{j=1}^m \sum_{t=1}^{n_j} c_{i^j_t}(s_{i^j_t}, s_{-i^j_t}) \\ &= \sum_{j=1}^m \sum_{i\in N_{t_j}} c_i(s_i, s_{-i}) \\ &= \sum_{i\in N} c_i(s_i, s_{-i}), \end{aligned}$$

implying that the network game with multiple types of players is a BBPNG.

(Necessity) If the conclusion was not true, then there would not exist a constant real-valued function $d : S_0^m \to \mathbb{R}$ such that the potential function of the FNG denoted by $p : S_0^m \to \mathbb{R}$ can be expressed as

$$p(s_{t_1}, \cdots, s_{t_m}) = \sum_{j=1}^m c_{t_j}(s_{t_1}, \cdots, s_{t_m}) + d(s_{t_1}, \cdots, s_{t_m}). \tag{16}$$

Consider strategy profiles that players of the same type take the same strategy. Without loss of generality, assume

$$s_i := s^*_{t_j} \in S_0, \quad \forall\ i \in N_{t_j}.$$

Then, the strategy profile on each hyperedge is the same. Denote it as $s^* = (s^*_{t_1}, s^*_{t_2}, \cdots, s^*_{t_m})$.

One the one hand, since the network game with multiple types of players is a BBPG, according to Definition 3.2, it can be calculated by (14) and (15) that

$$\begin{aligned} P(s) &= \sum_{i=1}^n c_i(s_i, s_{-i}) \\ &= \sum_{i=1}^n \sum_{e\in\mathcal{E}_i} c^g_{T_i}(s_i, s_{-e\backslash\{i\}}) \\ &= \sum_{j=1}^m \sum_{i\in N_{t_j}} \sum_{e\in\mathcal{E}_i} c^g_{t_j}(s^*) \\ &= |\mathcal{E}| \sum_{j=1}^m c^g_{t_j}(s^*). \end{aligned} \tag{17}$$

On the other hand, since the network game with multiple types of players is a potential game, inserting (16) into (13), we have

$$P(s) = \sum_{e\in\mathcal{E}} p^g(s^*) = |\mathcal{E}|[\sum_{j=1}^m c^g_{t_j}(s^*) + d(s^*)]. \tag{18}$$

Comparing (17) with (18), we obtain

$$d(s^*) = 0. \tag{19}$$

Since $s^* \in S_0^m$ is arbitrary, d is a constant function, which contracts with above assumption. ■

C. Budget-Balanced Potential Games

In this subsection, we search for a basis of $\mathcal{G}^{\mathcal{BP}}_{[n;k_1,\cdots,k_n]}$. For this, we need to prove that $\mathcal{G}^{\mathcal{BP}}_{[n;k_1,\cdots,k_n]}$ is a subspace of $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ by verifying that $\mathcal{G}^{\mathcal{BP}}_{[n;k_1,\cdots,k_n]}$ remains closed with respect to the addition and multiplication operations defined on $\mathcal{G}_{[n;k_1,\cdots,k_n]}$. The proof is straightforward, and hence, is omitted here.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3337688

In the following, we give a survey for the vector space structure of finite games. Consider a finite game $G = (N,(S_i)_{i\in N},(c_i)_{i\in N}) \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. Express $j \in S_i$ in its vector form, i.e., $\delta_{k_i}^j$. Then, according to Proposition 2.2, there is a unique row vector $V_i^c \in \mathbb{R}^k$, such that c_i can be expressed as

$$c_i(s_1,\cdots,s_n) = V_i^c \ltimes_{j=1}^n \vec{s}_j, \quad i=1,\cdots,n. \tag{20}$$

Putting all V_i^c, $i=1,\cdots,n$, together, we define

$$V_G := \left[V_1^c, V_c^2, \cdots, V_n^c\right] \in \mathbb{R}^{nk},$$

which is called the structure vector of G. Regarding a finite game as a point in $\mathbb{R}^{nk}$, it can be proved that $\mathcal{G}_{[n;k_1,\cdots,k_n]} \cong \mathbb{R}^{nk}$. Therefore, the analysis of finite games can be put into a linear algebraic frame.

The following lemmas are given to show the relationship between the algebraic structure of BBPGs and that of non-strategy games [5].

Lemma 3.8: A finite game $G = (N,(S_i)_{i\in N},(c_i)_{i\in N})$ is a BBPG, if and only if, for every $i \in N$, every $s_j \in S_j$, every $s_{i,j} \in S_{-\{i,j\}}$, and any $s_i, s_i' \in S_i$, we have

$$\sum_{j\neq i} c_j(s_j, s_i, s_{-\{i,j\}}) = \sum_{j\neq i} c_j(s_j, s_i', s_{-\{i,j\}}). \tag{21}$$

Here, $s_{-\{i,j\}}$ is the strategy profile of the players except for players i and j, and $c_j(s_j, s_i, s_{-\{i,j\}})$ is the utility of player j taking strategy s_j in the case of player i taking strategy s_i and other players taking strategy profile $s_{-\{i,j\}}$.

Proof: For every $i \in N$, every $s_{-i} \in S_{-i}$, and any $s_i \in S_i$, $p(s)$ in (4) can be expressed as

$$p(s_i, s_{-i}) = c_i(s_i, s_{-i}) + \sum_{j\neq i} c_j(s_j, s_i, s_{-\{i,j\}}). \tag{22}$$

Substituting (22) into (3) yields the necessity. The sufficiency is obtained by adding $(c_i(s_i, s_{-i}) + c_i(s_i', s_{-i}))$ to both side of (21) and using (22). ■

For any finite game $G = (N,(S_i)_{i\in N},(c_i)_{i\in N})$, define a corresponding game $\widetilde{G} = (N,(S_i)_{i\in N},(h_i)_{i\in N})$, where for every $i \in N$, every $s_{-i} \in S_{-i}$,

$$h_i(s_i, s_{-i}) = \sum_{j\neq i} c_j(s_j, s_i, s_{-\{i,j\}}), \quad \forall\ s_i \in S_i. \tag{23}$$

Then, the following result can be obtained, which is an immediate consequence of Lemma 3.8.

Lemma 3.9: A finite game $G = (N,(S_i)_{i\in N},(c_i)_{i\in N})$ is a BBPG, if and only if, its corresponding game $\widetilde{G}$ is a non-strategy game. That is, for every $i \in N$ and every $s_{-i} \in S_{-i}$,

$$h_i(s_i, s_{-i}) = h_i(s_i', s_{-i}), \quad \forall\ s_i, s_i' \in S_i.$$

According to Proposition 2.2, h_i and c_i can be expressed in vector forms, respectively, as

$$\begin{aligned} c_i(s_i, s_{-i}) &= V_i^c \ltimes_{t=1}^n \vec{s}_t, \quad i=1,2,\cdots,n; \\ h_i(s_i, s_{-i}) &= V_i^h \ltimes_{t=1}^n \vec{s}_t, \quad i=1,2,\cdots,n. \end{aligned}$$

Construct a matrix $Q \in \mathcal{M}_{nk\times nk}$ with $k = \prod_{i=1}^n k_i$ as below:

$$Q = \begin{bmatrix} \mathbf{0}_{k\times k} & I_{k\times k} & I_{k\times k} & \cdots & I_{k\times k} \\ I_{k\times k} & \mathbf{0}_{k\times k} & I_{k\times k} & \cdots & I_{k\times k} \\ I_{k\times k} & I_{k\times k} & \mathbf{0}_{k\times k} & \cdots & I_{k\times k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{k\times k} & I_{k\times k} & I_{k\times k} & \cdots & \mathbf{0}_{k\times k} \end{bmatrix}.$$

Using vector forms of c_i and h_i, (23) can be expressed in vector form as

$$\begin{aligned} V_i^h \ltimes_{t=1}^n \vec{s}_t &= \sum_{j\neq i} (V_j^c \ltimes_{t=1}^n \vec{s}_t) \\ &= (\sum_{j\neq i} V_j^c) \ltimes_{t=1}^n \vec{s}_t \\ &= [V_1^c, \cdots, V_n^c] Q_i^{\mathrm{T}} \ltimes_{t=1}^n \vec{s}_t, \end{aligned} \tag{24}$$

where Q_i is $i-th$ row block of Q. Since $s_1, s_2, \cdots, s_n$ are arbitrary, the following matrix equation holds from (24):

$$V_i^h = [V_1^c, \cdots, V_n^c] Q_i^{\mathrm{T}}.$$

Putting all V_i^h, $i=1,2,\cdots,n$, together and taking transpose, we obtain the following liner system:

$$[V_1^h, \cdots, V_n^h]^{\mathrm{T}} = Q[V_1^c, \cdots, V_n^c]^{\mathrm{T}}.$$

It can be verified that Q is invertible, and

$$Q^{-1} = \begin{bmatrix} -\frac{n-2}{n-1} I_k & \frac{1}{n-1} I_k & \cdots & \frac{1}{n-1} I_k \\ \frac{1}{n-1} I_k & -\frac{n-2}{n-1} I_k & \cdots & \frac{1}{n-1} I_k \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} I_k & \frac{1}{n-1} I_k & \cdots & -\frac{n-2}{n-1} I_k \end{bmatrix}.$$

Hence, we have

$$V_G^{\mathrm{T}} = Q^{-1} V_{\widetilde{G}}^{\mathrm{T}}. \tag{25}$$

(25) shows that the subspace of BBPGs is isomorphic to that of non-strategy games. Results about spatial structure of non-strategy games have been given in [5]. Denote

$$E_i = I_{\prod_{t=1}^{i-1} k_t} \otimes \mathbf{1}_{k_i} \otimes I_{\prod_{t=i+1}^{n} k_t} \in \mathcal{M}_{k\times \frac{k}{k_i}}, \quad i=1,2,\cdots,n.$$

Particularly, we specify, when $i=1$, $I_{\prod_{t=1}^{0} k_t} = 1$; when $i=n$, $I_{\prod_{t=n+1}^{n} k_t} = 1$. Define

$$B_N = \begin{bmatrix} E_1 & \mathbf{0}_{k\times\frac{k}{k_2}} & \cdots & \mathbf{0}_{k\times\frac{k}{k_n}} \\ \mathbf{0}_{k\times\frac{k}{k_1}} & E_2 & \cdots & \mathbf{0}_{k\times\frac{k}{k_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{k\times\frac{k}{k_1}} & \mathbf{0}_{k\times\frac{k}{k_2}} & \cdots & E_n \end{bmatrix} \in \mathcal{M}_{nk\times \sum\limits_{i=1}^{n}\frac{k}{k_i}}.$$

Denote by $\mathcal{N}$ the subspace of non-strategy games. Then, we have $dim(\mathcal{N}) = \sum\limits_{i=1}^{n} \frac{k}{k_i}$, and $\mathcal{N} \in Span(B_N)$, which has $\mathrm{Col}(B_N)$ as its basis. Applying these two results to (25), we obtain the following result.

Theorem 3.10: 1) The dimension of subspace of $\mathcal{G}_{[n;k_1,\cdots,k_n]}^{\mathcal{BP}}$ is

$$dim(\mathcal{G}_{[n;k_1,\cdots,k_n]}^{\mathcal{BP}}) = \sum_{i=1}^{n} \frac{k}{k_i}.$$

2) The subspace of $\mathcal{G}_{[n;k_1,\cdots,k_n]}^{\mathcal{BP}}$ is

$$\mathcal{G}_{[n;k_1,\cdots,k_n]}^{\mathcal{BP}} \in Span(\Psi),$$

where

$$\Psi := Q^{-1} B_N = \begin{bmatrix} -\frac{n-2}{n-1} E_1 & \frac{1}{n-1} E_2 & \cdots & \frac{1}{n-1} E_n \\ \frac{1}{n-1} E_1 & -\frac{n-2}{n-1} E_2 & \cdots & \frac{1}{n-1} E_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} E_1 & \frac{1}{n-1} E_2 & \cdots & -\frac{n-2}{n-1} E_n \end{bmatrix}. \tag{26}$$

Moreover, the subspace of $\mathcal{G}_{[n;k_1,\cdots,k_n]}^{\mathcal{BP}}$ has $\mathrm{Col}(\Psi)$ as a basis.

IV. Optimization of Multi-Agent Systems via Designed Utilities

A multi-agent system with multiple types of agents is a tuple $\prod = (T, (S_i)_{i\in N}, (N, \mathcal{E}), (f_j)_{j\in T}, W)$. Here, $N := \{1, 2, \cdots, n\}$ is the set of finite agents, and every agent i belongs to a certain type denoted by T_i. $T := \{t_1, t_2, \cdots, t_m\}$ is the set of all types. Given that agents of the same type tend to have the same performance, we assume the cardinalities of their strategy sets are the same, and denote by $S_{t_j} = \{1, 2, \cdots, k_{t_i}\}$ the strategy set of players of type t_j, $j = 1, 2, \cdots, m$. $(N, \mathcal{E})$ is an $m-$uniform hypergraph with each hyperedge including m different types of agents. Only agents on the same hyperedge can communicate, and each agent obtains some utility after the communication. The specific utility of players of type j is described by f_j, $j = 1, 2, \cdots, m$, which are called type-based utility functions. Assume an agent that belongs to more than one hyperedge communicates on each hyperedge, respectively, with the same strategy. A system objective function $W : S \to \mathbb{R}$ is preassigned, representing the overall utility of the whole network. Our purpose is to minimize W, namely, to find a profile $s_* \in S$, such that

$$W(s_*) = \min_{s\in S} W(s).$$

We model the multi-agent system with multiple types of agents as a network game with multiple types of players by specifying the FNG and the utility functions of players. The FNG $g := (N_g, (S^g_j)_{j\in N_g}, (c^g_j)_{j\in N_g})$ is defined as below:

1) $N_g = T := \{t_1, t_2, \cdots, t_m\}$;
2) $S^g_j = S_{t_j} := \{1, 2, \cdots, k_{t_j}\}, \quad j = 1, 2, \cdots, m$;
3) $c^g_j := f_j, \quad j = 1, 2, \cdots, m.$

The utility function of player i is defined as below:

$$c_i(s_i, s_{-i}) = \sum_{e\in\mathcal{E}_i} c^g_{T_i}(s_i, s_{e\setminus\{i\}}), \quad i = 1, 2, \cdots, n. \quad (27)$$

Then, for the system objective function $W(s)$ we have:

$$W(s) = \sum_{i=1}^{n} c_i(s), \quad \forall\ s \in S.$$

The fundamental idea of the technique developed in this paper is: choosing suitable type-based utility functions such that the multi-agent system with multiple types of agents becomes a BBPNG with $W(s)$ as its potential function.

In the following, we construct a linear system which turns the problem of designing suitable type-based utility functions into checking whether a solution of this linear system exists.

First, we express (13) in its algebraic form. Consider a hyperedge $e = \{i_{t_1}, i_{t_2}, \cdots, i_{t_m}\}$ with i_{t_j} the player of type t_j, $j = 1, 2, \cdots, m$. For the right side of (13), p^g can be expressed in vector form as:

$$p^g(s_{t_1}, s_{t_2}, \cdots, s_{t_m}) = V_{p^g} \ltimes_{j=1}^{m} \vec{s}_{t_j}. \quad (28)$$

For the left side of (13), p can be expressed in vector form as:

$$p(s_1, s_2, \cdots, s_n) = V_p \ltimes_{i=1}^{n} \vec{s}_i. \quad (29)$$

By comparing (28) and (29), it can be found that there are m strategies multiplied in (28), while n strategies in (29). We define a deleting operator as below to convert the vector form in (28) into that in (29):

$$E_e = \otimes_{j=1}^{m}\Gamma_j, \quad j = 1, 2, \cdots, t, \quad (30)$$

where

$$\Gamma_j = \begin{cases} I_\kappa, & j \in e; \\ \mathbf{1}^{\mathrm{T}}_\kappa, & j \notin e. \end{cases}$$

Then, the following properties can be verified:

$$E_e \ltimes_{i=1}^{n} \vec{s}_i = \ltimes_{j=1}^{m} \vec{s}_{t_j}. \quad (31)$$

Substituting (28) and (29) into (13), by (31) we obtain,

$$\begin{aligned} V_p \ltimes_{i=1}^{n} \vec{s}_i &= \sum_{e\in\mathcal{E}} V_{p^g} \ltimes_{j=1}^{m} \vec{s}_{t_j} \\ &= \sum_{e\in\mathcal{E}} V_{p^g} E_e \ltimes_{i=1}^{n} \vec{s}_i \\ &= V_{p^g}(\sum_{e\in\mathcal{E}} E_e) \ltimes_{i=1}^{n} \vec{s}_i. \end{aligned}$$

Denote

$$E = \sum_{e\in\mathcal{E}} E_e.$$

Since s_i, $i = 1, 2, \cdots, n$, are arbitrary, we have

$$V_p = V_{p^g} E. \quad (32)$$

Theorem 4.1: Let $W(s)$ be the given system objective function of a multi-agent system with multiple types of agents. Type-based utility functions can be found such that the multi-agent system becomes a BBPNG with W as its potential function, if and only if, the following linear systems has a solution for unknown ζ:

$$V^{\mathrm{T}}_w = E^{\mathrm{T}}(\mathbf{1}^{\mathrm{T}}_n \otimes I_{\kappa^n})\Psi\zeta, \quad (33)$$

where Ψ is defined in (26). Moreover, type-based utility functions can be designed as $\Psi\zeta$.

Proof: On the one hand, according to Theorem 3.10, we have, for unknown ζ,

$$V^{\mathrm{T}}_g = \Psi\zeta. \quad (34)$$

On the other hand, according to Definition 3.2, it can be checked that

$$V_{p^g} = \sum_{j=1}^{m} V_{t_j} = V_g(\mathbf{1}_n \otimes I_{\kappa^n}). \quad (35)$$

Inserting (35) into (32), we have

$$V_p = V_g(\mathbf{1}_n \otimes I_{\kappa^n})E. \quad (36)$$

Substituting (34) into (36), replacing V_p with V_w, and taking transpose, we obtain (33). The design method of type-based utility functions follows from (34). ■

Example 4.2: Consider a multi-agent system with 5 agents and 3 types. Assume $|S_i| = 2$, $i = 1, \cdots, 5$, $T = \{t_1 := \bullet, t_2 := \blacktriangle, t_3 := \blacksquare\}$, the hypergraph is as shown in Figure 3, and the system objective function W is as given in Table I.

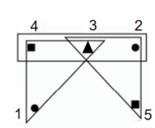

Fig. 3. 3-uniform hypergraph

TABLE I

SYSTEM OBJECTIVE FUNCTION W:

s	11111	11112	11121	11122	11211	11212	11221
W	18	17	16	15	24	24.5	25
s	11222	12111	12112	12121	12122	12211	12212
W	25.5	26	23.5	22	22	26	26
s	12221	12222	21111	21112	21121	21122	21211
W	26.5	26.5	22	21	19.5	18.5	25
s	21212	21221	21222	22111	22112	22121	22122
W	25.5	25.5	26	30	28.5	27	25.5
s	22211	22212	22221	22222			
W	27	27	27	27			

Using (30), we have

$$
\begin{aligned}
E_{e_1} &= I_2 \otimes \mathbf{1}_2 \otimes I_2 \otimes I_2 \otimes \mathbf{1}_2; \\
E_{e_2} &= \mathbf{1}_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \mathbf{1}_2; \\
E_{e_3} &= \mathbf{1}_2 \otimes I_2 \otimes I_2 \otimes \mathbf{1}_2 \otimes I_2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E &= E_{e_1} + E_{e_2} + E_{e_3} \\
&= \left[\begin{smallmatrix}
3&2&1&0&0&0&0&0&1&1&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&2&3&0&0&0&0&0&0&1&1&0&0&0&0&0&1&1&2&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&3&2&1&0&0&0&0&0&1&1&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&2&3&0&0&0&0&0&0&1&1&0&0&0&0&0&1&1&2&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&1&1&0&0&0&0&0&0&3&2&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&2&0&0&0&0&0&0&1&1&0&0&0&0&0&1&2&3&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&1&1&0&0&0&0&0&0&3&2&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&0&0&0&0&0&0&1&1&0&0&0&0&0&1&2&3
\end{smallmatrix}\right].
\end{aligned}
$$

Using (26), it can be calculated that

$$
\Psi = \begin{bmatrix} -A & B & C \\ A & -B & C \\ A & B & -C \end{bmatrix},
$$

where

$$
\begin{aligned}
A &= \tfrac{1}{2}(\mathbf{1}_2 \otimes I_4); \\
B &= \tfrac{1}{2}(I_2 \otimes \mathbf{1}_2 \otimes I_2); \\
C &= \tfrac{1}{2}(I_4 \otimes \mathbf{1}_2).
\end{aligned}
$$

Inserting E, Ψ and $W(s)$ into (33), it can be verified that (33) has solutions. For instance, one of the solutions is

$$
\zeta = [4, 3, 5, 6, 5, 4, 8, 6, 3, 6, 8, 5].
$$

According to Theorem 4.1, we obtain the type-based utility functions shown in Table II.

TABLE II

UTILITY MATRIX:

$c\backslash s$	111	112	121	122	211	212	221	222
$c^g_{t_1}(\bullet)$	2	2	3	1.5	6	5.5	4	2
$c^g_{t_2}(\blacktriangle)$	1	1	3	4.5	2	2.5	1	3
$c^g_{t_3}(\blacksquare)$	3	2	2	2.5	2	0.5	4	4

V. CONCLUSION

Optimization of the multi-agent system with multiple types of agents has been studied via potential game approach. First, the verification of BBPNGs has been simplified to verify whether the FNG is a BBPG. Then, the algebraic expression of BBPGs has been given, showing that the subspace of BBPGs is isomorphic to that of non-strategic games. By resorting to the structure of non-strategic games, a basis of subspace of BBPGs has been obtained as well as the space dimension. Finally, by expressing the potential function of a BBPNG in matrix form, a linear system has been presented, which transforms the problem of designing type-based utility functions into the existence problem of solutions of this linear system.

The cardinality of all hyperedges and that of the strategy sets of the players of the same type are both assumed to be the same in this paper. Moderately relaxing these two conditions will lead to new network game models, such as that with multiple FNGs. Then, the assumptions in the proof of main results no longer hold, which calls for new research methods and techniques.

REFERENCES

[1] A. Nedić, A. Ozdaglar, "Cooperative distributed multi-agent optimization", in Daniel P. Palomar, Yonina C. Eldar (Eds.), *Convex Optimization in Signal Processing and Communications*, pp. 340-386, Cambridge, Cambridge University Press, 2010.

[2] F. Battison et al., "Networks beyond pairwise interactions: structure and dynamics", *Physics Reports*, vol. 874, pp. 1-92, 2020.

[3] S. Boyd, L.Vandenberghe, *Convex Optimization*, Cambridge, Cambridge University Press, 2004.

[4] D. Cheng, H. Qi, Y. Zhao, *An Introduction to Semi-tensor Product of Matrices and Its Applications*, World Scientific, Singapore, 2012.

[5] D. Cheng, T. Liu, K. Zhang, H. Qi, "On decomposed subspaces of finite games", *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3651-3656, Nov., 2016.

[6] R. Gopalakrishnan, J.R. Marden, A. Wierman, "An architectural view of game theoretic control", *Performance Evaluation Review*, vol. 38, no. 3, pp. 31-36, Jan., 2011.

[7] R. Gopalakrishnan, J.R. Marden, A. Wierman, "Potential games are necessary to ensure pure Nash equilibria in cost sharing games", *Mathelmatics of Operations Research*, vol. 39, no. 4, pp. 1252-1296, Feb., 2014.

[8] S. H. Hwang, L.Rey-Bellet, "Strategic decompositions of normal form games: zero-sum games and potential games", *Games and Economic Behavior*, vol. 122, pp. 370-390, 2020.

[9] Q.D. Lä, Y.H. Chew, B. Soong, *Potential Game Theory: Applications in Radio Resource Allocation*, New York, Springer, 2016.

[10] A. Liu and H. Li, "Matrix approach to verification of finite multi-potential games," *Journal of the Franklin Institute*, vol. 359, pp. 2229-2243, 2022.

[11] T. Liu, J. Wang, X. Zhang, D. Cheng, "Game theoretic control of multi-agent systems", *SIAM Journal on Control and Optimization*, vol. 57, pp. 1691-1709, Jul., 2019.

[12] J.R. Marden, A. Wierman, "Distributed welfare games", *Operations Research*, vol. 61, no. 1, pp. 155-168, Jan., 2013.

[13] D. Monderer, L.S. Shapley, "Potential Games", *Games and Economic Behavior*, vol. 14, no. 1, pp. 124-143, May, 1996.

[14] F. Yousefian, A. Nedić, U.V. Shanbhag et al., "Self-Tuned Stochastic Approximation Schemes for Non-Lipschitzian Stochastic Multi-User Optimization and Nash Games", *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1753-1766, Jul. 2016.

[15] P.Y. Zhang, C.X. Li, M.C. Zhou, W.J. Huang, "Transaction transmission model for blockchain channels based on non-cooperative games", *Science China Information Sciences*, vol. 66, no. 1, pp. 112105:1-112105:18, Jan., 2023.

[16] M.H. Zhu, S. Martinez, "Distributed coverage games for energy-auaremobile sensor networks", *SIAM Journal on Control and Optimizaton*, vol. 51, no. 1, pp. 1-27, Oct. 2013.